

A resolvent-type method for estimating time integrals of quadratic fluctuations in weakly asymmetric exclusion

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Abstract

For general right processes η_s in a stationary state ν , under fairly weak conditions, it is shown that $\int_0^T dt \mathbf{E}_\nu [\int_0^t V(s, \eta_{s\varepsilon^{-\kappa}}) ds]^2 \leq c_T \int_0^T (V(s, \cdot) | (1 - \varepsilon^{-\kappa} L^{\text{sym}})^{-1} V(s, \cdot)) ds$ where L^{sym} denotes the symmetric part of the generator of η_s on the Hilbert space $L^2(\nu)$ with inner product $(\cdot | \cdot)$. Compared to $\int_0^T (V(s, \cdot) | (-\varepsilon^{-\kappa} L^{\text{sym}})^{-1} V(s, \cdot)) ds$ which is often used in this context, the advantage is that $(1 - \varepsilon^{-\kappa} L^{\text{sym}})^{-1} V$ always exists for bounded measurable V . As a consequence one obtains useful estimates of time integrals of ε -scaled quadratic fluctuations V_ε build from $V_\#(\eta) = (\eta(0) - 1/2)(\eta(1) - 1/2)$ in the case of $\sqrt{\varepsilon}$ -asymmetric exclusion with ν being the symmetric Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}}$.

KEY WORDS right process, resolvent, generator, scaling limit, replacement lemma

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1 Motivation and Summary

Assume that a particle system $\eta_s(x)$, $x \in \mathbb{Z}^d$, $s \geq 0$, gives raise to a scaled field of type

$$Y_s^\varepsilon = \varepsilon^{\lambda_d} \sum_x \frac{\eta_{s\varepsilon^{-\kappa}}(x) - a}{\chi} \delta_{\varepsilon x - bs\varepsilon^{-\tilde{\kappa}}}, \quad s \geq 0, \quad (1.1)$$

where $\delta_{\varepsilon x - bs\varepsilon^{-\tilde{\kappa}}}$ denotes the Dirac measure concentrated in the macroscopic point $\varepsilon x - bs\varepsilon^{-\tilde{\kappa}}$ and one wishes to understand the limiting behaviour, $\varepsilon \downarrow 0$, of this field. Usually it follows from the martingale problem for the strong Markov process $\eta_s(x)$ that there is an approximate equation for Y_s^ε , ε small, which often reads like

$$dY_s^\varepsilon(G) \sim Y_s^\varepsilon(\mathcal{A}G) ds + V_\varepsilon^G(s, \xi_{s\varepsilon^{-\kappa}}) ds + dM_s^{G, \varepsilon} \quad (1.2)$$

where G is a smooth test function on \mathbb{R}^d with compact support, \mathcal{A} a partial differential operator, V_ε^G is for fixed ε , G a bounded measurable function, $\xi_{s\varepsilon^{-\kappa}}$ stands for $(\eta_{s\varepsilon^{-\kappa}}(x) - a)/\chi$

and $M^{G,\varepsilon}$ denotes a martingale. For further analysis of this approximate equation it becomes necessary to express $V_\varepsilon^G(s, \xi_{s\varepsilon-\kappa})$ in terms of the field Y_s^ε , that is, one wants to replace

$$\int_0^t V_\varepsilon^G(s, \xi_{s\varepsilon-\kappa}) ds \quad \text{by} \quad \int_0^t F(s, Y_s^\varepsilon, G) ds$$

in some sense where the functional F has to be found (see [KL1999] for a good review of this method). Typically the difference

$$\int_0^t F(s, Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(s, \xi_{s\varepsilon-\kappa}) ds \quad \text{simplifies to} \quad \sum_{i=1}^m \int_0^t V_\varepsilon^{G,i}(s, \xi_{s\varepsilon-\kappa}) ds$$

and the task is to estimate an appropriate norm of $t \mapsto \int_0^t V_\varepsilon(s, \eta_{s\varepsilon-\kappa}) ds$ where $V_\varepsilon(s, \eta)$ stands for one of the functions $V_\varepsilon^{G,i}(s, (\eta - a)/\chi)$, $i = 1, \dots, m$.

First, for an arbitrary but fixed $\beta > 0$ and a finite time horizon T , it follows from Lemma 2.1 in Section 2 that

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V_\varepsilon(s, \eta_{s\varepsilon-\kappa}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta} \varepsilon^{2\kappa} (\tilde{V}_\varepsilon | G_{\frac{\beta}{2}\varepsilon^\kappa} \tilde{V}_\varepsilon)_{L^2(ds \otimes d\nu)} \quad (1.3)$$

where \tilde{V}_ε stands for the function $(s, \eta) \mapsto e^{-\frac{\beta}{2}s\varepsilon^\kappa} V_\varepsilon(s\varepsilon^\kappa, \eta)$ and $(G_\alpha)_{\alpha>0}$ denotes the strongly continuous contraction resolvent associated with the process $(s, \eta_s)_{s \geq 0}$ on the Hilbert space $H = L^2(ds \otimes d\nu)$ assuming that there exists an invariant state ν of the system $(\eta_s)_{s \geq 0}$. Notice that (1.3) is valid in the context of general right processes.

The observation is now that in many cases one has the inequality

$$(u | G_\alpha u) \leq (u | (\alpha - L_s)^{-1} u), \quad u \in H, \quad (1.4)$$

for all $\alpha > 0$ by abstract theory on resolvents where L_s stands for the symmetric part of the generator of the resolvent $(G_\alpha)_{\alpha>0}$ in H .

Second, choosing $\alpha = \frac{\beta}{2}\varepsilon^\kappa$ and $u = \tilde{V}_\varepsilon$ in (1.4) and applying (1.3) yields

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V_\varepsilon(s, \eta_{s\varepsilon-\kappa}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta} \varepsilon^\kappa \int_0^T (V_\varepsilon(s, \cdot) | (\frac{\beta\varepsilon^\kappa}{2} - L^{\text{sym}})^{-1} V_\varepsilon(s, \cdot))_{L^2(\nu)} ds \quad (1.5)$$

where L^{sym} denotes the symmetric part of the generator of the process $(\eta_s)_{s \geq 0}$ in $L^2(\nu)$. The details of how to replace L_s by L^{sym} are explained by Lemma 2.8 in Section 2.

Remark that applying Kipnis-Varadhan's inequality¹ which is widely used in the context of particle systems would yield

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V_\varepsilon(s, \eta_{s\varepsilon-\kappa}) ds \right]^2 \leq 14T \varepsilon^\kappa \int_0^T (V_\varepsilon(s, \cdot) | (-L^{\text{sym}})^{-1} V_\varepsilon(s, \cdot))_{L^2(\nu)} ds$$

instead of (1.5). But, in the important case where $(\eta_s)_{s \geq 0}$ is a simple one-dimensional exclusion process and ν is the symmetric Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}}$, the right-hand side of the last inequality is infinite for $V_\#(\eta) = (\eta(0) - 1/2)(\eta(1) - 1/2)$ which is

¹See [CLO2001, Lemma 4.3] for the version used here.

the quadratic part of the normalised current and the most basic quadratic fluctuation. So Kipnis-Varadhan's inequality cannot be used to estimate time integrals of ε -scaled quadratic fluctuations V_ε build from $V_\#$.

However, the right-hand side of the inequality (1.5) is always finite for bounded measurable functions V_ε . Hence, when substituting $V_\varepsilon^{G,i}$ for V_ε , $i = 1, \dots, m$, this inequality gives a tool for how to show

$$\lim_{\varepsilon \rightarrow 0} \int_0^T dt \mathbf{E}_\nu \left(\int_0^t F(s, Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(s, \xi_{s\varepsilon^{-\kappa}}) ds \right)^2 = 0. \quad (1.6)$$

Remark that replacing $\int_0^t V_\varepsilon^G(s, \xi_{s\varepsilon^{-\kappa}}) ds$ by $\int_0^t F(s, Y_s^\varepsilon, G) ds$ in the sense of the above limit is rather weak since the replacement does not hold for every $t \in [0, T]$ but only for an average over $t \in [0, T]$. Nevertheless, as recently shown in [A2012], this weak form of a replacement is still sufficient for deriving a meaningful equation which could be used as the limit of (1.2).

Section 2 presents the resolvent method which is based on (1.3),(1.4),(1.5). A detailed proof of the inequality (1.4) is given in a general setting (see Corollary 2.6). The result as such cannot be new. However, the author could not find a reference. The proof is based on a variational formula (see Lemma 2.4) which was also used in the proof of [LQSY2004, Lemma 2.1] but without explicit proof and only in the framework of exclusion processes. The detailed proof is added for completeness and for having a good account on the precise conditions needed.

In Section 3 the resolvent method is applied in the case of $\sqrt{\varepsilon}$ -asymmetric one-dimensional simple exclusion in equilibrium. The field (1.1) of interest is the diffusively scaled density fluctuation field. In the corresponding equation (1.2), \mathcal{A} is the one-dimensional Laplacian and the bounded functions V_ε^G are ε -scaled quadratic fluctuations build from $V_\#$ introduced above. The key result is Lemma 3.3 which leads to a replacement of the time integrals of these quadratic fluctuations in the sense of (1.6), see Corollary 3.4. Remark that the density fluctuations in $\sqrt{\varepsilon}$ -asymmetric exclusion are related to non-trivial distributions like the Tracy-Widom distribution hence they are a good 'medium' for testing techniques. Proving Lemma 3.3 using a resolvent-type method can be considered to be such a test.

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2 A Resolvent Method

Let X be a Hausdorff topological space and assume that the Borel- σ -algebra on X is equal to the σ -algebra generated by the set of all continuous functions on X .

Denote by $(\Omega, \mathcal{F}, \mathbf{P}_\eta, \eta \in X, (\eta_s)_{s \geq 0})$ a right process with state space X , infinite life time and corresponding filtration \mathcal{F}_s , $s \geq 0$, satisfying the usual conditions (see [S1988] for a good account on general right processes). Assume that there exists a measure ν on X which is an invariant state of this right process and denote by \mathbf{E}_ν the expectation operator given by the probability measure $\int_X \mathbf{P}_\eta \nu(d\eta)$.

Obviously, the pair $(s, \eta_s)_{s \geq 0}$ gives another right process $(\Omega, \mathcal{F}, \mathbf{P}_{s,\eta}, (s, \eta) \in [0, \infty) \times X, (s, \eta_s)_{s \geq 0})$ corresponding to the same filtration \mathcal{F}_s , $s \geq 0$, such that

$$\mathbf{P}_{s,\eta}(\{\eta_s = \eta\}) = 1 \quad \text{for all} \quad (s, \eta) \in [0, \infty) \times X.$$

Define the transition semigroup and the resolvent of $(s, \eta_s)_{s \geq 0}$ by

$$p_r V(s, \eta) = \mathbf{E}_{s, \eta} V(s + r, \eta_{s+r}), \quad r \geq 0,$$

and

$$R_\alpha V(s, \eta) = \int_0^\infty dr e^{-\alpha r} p_r V(s, \eta), \quad \alpha > 0,$$

respectively, where V is an arbitrary bounded measurable function on $[0, \infty) \times X$.

Denote by ℓ the Lebesgue measure on $[0, \infty)$ and notice that $\ell \otimes \nu$ is an excessive measure on $[0, \infty) \times X$ with respect to $(\Omega, \mathcal{F}, \mathbf{P}_{s, \eta}, (s, \eta) \in [0, \infty) \times X, (s, \eta_s)_{s \geq 0})$ because

$$\int p_r V d(\ell \otimes \nu) \leq \int V d(\ell \otimes \nu), \quad r \geq 0,$$

for all non-negative measurable functions V on $[0, \infty) \times X$. As a consequence, see Section IV.2 in [MR1992] for the details, there exists a strongly continuous contraction resolvent $(G_\alpha)_{\alpha > 0}$ on $L^2(\ell \otimes \nu)$ such that $G_\alpha V$ is an $(\ell \otimes \nu)$ -version of $R_\alpha V$ for all $\alpha > 0$ and all bounded functions V in $L^2(\ell \otimes \nu)$.

Lemma 2.1 *Fix $\beta > 0$, let $T > 0$ be a finite time horizon and consider the process $(\eta_{cs})_{s \geq 0}$ time-scaled by a factor $c > 0$. Then*

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V(s, \eta_{cs}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta c^2} (\tilde{V} | G_{\frac{\beta}{2c}} \tilde{V})_{L^2(\ell \otimes \nu)}$$

for all bounded functions V in $L^2(\ell \otimes \nu)$ where $\tilde{V}(s, \eta) = e^{-\frac{\beta}{2}s/c} V(s/c, \eta)$.

Proof. Choose a bounded function V in $L^2(\ell \otimes \nu)$ and set $\hat{V}(s, \eta) = V(s/c, \eta)$. Then:

$$\begin{aligned} \int_0^T dt \mathbf{E}_\nu \left[\int_0^t V(s, \eta_{cs}) ds \right]^2 &\leq e^{\beta T} \int_0^\infty dt e^{-\beta t} \mathbf{E}_\nu \left[\int_0^t \hat{V}(cs, \eta_{cs}) ds \right]^2 \\ &= \frac{2e^{\beta T}}{c^2} \int_0^\infty dt e^{-\beta t} \mathbf{E}_\nu \int_0^{ct} ds \hat{V}(s, \eta_s) \int_s^{ct} dr \hat{V}(r, \eta_r) \\ &= \frac{2e^{\beta T}}{c^2} \int_0^\infty dt e^{-\beta t} \int_0^{ct} ds \int_s^{ct} dr \mathbf{E}_\nu \hat{V}(s, \eta_s) p_{r-s} \hat{V}(s, \eta_s) \\ &= \frac{2e^{\beta T}}{c^2} \int_0^\infty dt e^{-\beta t} \int_0^{ct} ds \int_0^{ct-s} dr \int d\nu \hat{V}(s, \cdot) p_r \hat{V}(s, \cdot) \\ &= \frac{2e^{\beta T}}{\beta c^2} \int_0^\infty ds \int_0^\infty dr e^{-\beta(s+r)/c} \int d\nu \hat{V}(s, \cdot) p_r \hat{V}(s, \cdot) \\ &= \frac{2e^{\beta T}}{\beta c^2} \int_0^\infty ds \int_0^\infty dr e^{-\beta(s+r)/c} \mathbf{E}_\nu \hat{V}(s, \eta_s) \hat{V}(s+r, \eta_{s+r}) \\ &= \frac{2e^{\beta T}}{\beta c^2} \int_0^\infty ds \int_0^\infty dr e^{-\frac{\beta}{2c}r} \mathbf{E}_\nu \tilde{V}(s, \eta_s) \tilde{V}(s+r, \eta_{s+r}) \\ &= \frac{2e^{\beta T}}{\beta c^2} \int_0^\infty ds \int d\nu \tilde{V}(s, \cdot) \int_0^\infty dr e^{-\frac{\beta}{2c}r} p_r \tilde{V}(s, \cdot) = \frac{2e^{\beta T}}{\beta c^2} (\tilde{V} | G_{\frac{\beta}{2c}} \tilde{V})_{L^2(\ell \otimes \nu)} \quad \blacksquare \end{aligned} \tag{2.1}$$

Remark 2.2 (i) In the case where V is time independent it easily follows from (2.1) that

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V(\eta_{cs}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta^2 c} (V | G_{\beta/c} V)_{L^2(\nu)}$$

because $V = \hat{V}$ and $\int_0^\infty ds e^{-\beta s/c} = c/\beta$.

(ii) Notice that one cannot apply (i) to the process $(s, \eta_s)_{s \geq 0}$ because $\ell \otimes \nu$ is not an invariant measure for this process.

The remaining part of this section deals with the problem of estimating the right-hand side of the inequality in Lemma 2.1 by something which is more likely to be computable in an explicit way.

Let $(G_\alpha)_{\alpha > 0}$ be a strongly continuous contraction resolvent on a real Hilbert space H with inner product $(\cdot | \cdot)$. If G_α^* denotes the adjoint of G_α then $(G_\alpha^*)_{\alpha > 0}$ is also a strongly continuous contraction resolvent on H . Denote by $(L, D(L))$ and $(L^*, D(L^*))$ the generators of $(G_\alpha)_{\alpha > 0}$ and $(G_\alpha^*)_{\alpha > 0}$, respectively.

Remark 2.3 If the co-generator $(L^*, D(L^*))$ is not the adjoint of $(L, D(L))$ then L^* coincides with the adjoint of $(L, D(L))$ on $D(L^*)$, at least.

The following assumption will play a crucial role in the proof of Lemma 2.4 below.

There exists $D_0 \subseteq H$ which is a core for both $(L, D(L))$ and $(L^*, D(L^*))$. (A1)

Remark that there are unbounded operators $(L, D(L))$ on Hilbert spaces whose adjoints $(L^*, D(L^*))$ satisfy $D(L) \cap D(L^*) = \{0\}$ in the worst case hence there is something to check for this assumption to hold.

Assuming (A1), the symmetric and antisymmetric parts of L given by

$$L_s = \frac{L + L^*}{2} \quad \text{and} \quad L_a = \frac{L - L^*}{2} \quad \text{respectively}$$

are defined on D_0 . Of course D_0 is dense in H since it is a core. As $(L, D(L))$ and $(L^*, D(L^*))$ are both negative definite, (L_s, D_0) is a symmetric, negative definite, densely defined operator. Hence it can be extended (Friedrich's extension for example) to a self-adjoint negative definite operator $(L_s, D(L_s))$ on H . Every such extension generates a strongly continuous contraction resolvent $((\alpha - L_s)^{-1})_{\alpha \geq 0}$ of self-adjoint positive definite operators on H .

Lemma 2.4 *Assume (A1) and fix both $\alpha > 0$ as well as an arbitrary self-adjoint extension of (L_s, D_0) . Then*

$$(u | G_\alpha u) = \sup_{v \in D_0} \left\{ 2(u | v) - ((\alpha - L)v | (\alpha - L_s)^{-1}(\alpha - L)v) \right\}$$

for all $u \in H$.

Proof. Fix $u \in H$. As D_0 is a core for both, $(L, D(L))$ and $(L^*, D(L^*))$, one can choose sequences $(v_n)_{n=1}^\infty$ and $(v_n^*)_{n=1}^\infty$ in D_0 such that

$$v_n \rightarrow \frac{G_\alpha}{2}u, \quad (\alpha - L)v_n =: \frac{u_n}{2} \rightarrow \frac{u}{2}, \quad v_n^* \rightarrow \frac{G_\alpha^*}{2}u, \quad (\alpha - L^*)v_n^* =: \frac{u_n^*}{2} \rightarrow \frac{u}{2},$$

in H when $n \rightarrow \infty$. Then

$$\left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(\alpha - L)(v_n + v_n^*) \right) \longrightarrow (u \mid G_\alpha u), \quad n \rightarrow \infty.$$

Indeed

$$\begin{aligned} & \left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(\alpha - L)(v_n + v_n^*) \right) \\ = & \left(\frac{\alpha - L}{2}(v_n + v_n^*) \mid (\alpha - L_s)^{-1}u_n \right) + \left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1} \frac{\alpha - L^*}{2} 2v_n^* \right) \end{aligned}$$

where

$$\begin{aligned} & \left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1} \frac{\alpha - L^*}{2} 2v_n^* \right) \\ = & \left((\alpha - L)(v_n + v_n^*) \mid 2v_n^* \right) - \left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1} \frac{\alpha - L^*}{2} 2v_n^* \right) \\ = & \left((v_n + v_n^*) \mid u_n^* \right) - \left(\frac{\alpha - L}{2}(v_n + v_n^*) \mid (\alpha - L_s)^{-1}u_n^* \right) \end{aligned}$$

hence

$$\begin{aligned} & \left((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(\alpha - L)(v_n + v_n^*) \right) \\ = & \left((v_n + v_n^*) \mid u_n^* \right) + \left(\frac{\alpha - L}{2}(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right). \end{aligned}$$

Of course

$$\left((v_n + v_n^*) \mid u_n^* \right) = \left(u_n^* \mid (v_n + v_n^*) \right) \longrightarrow (u \mid G_\alpha u), \quad n \rightarrow \infty,$$

whereas

$$\begin{aligned} \left(\frac{\alpha - L}{2}(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right) &= \left(\frac{\alpha - L}{2}v_n \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right) \\ &- \left(\frac{\alpha - L^*}{2}v_n^* \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right) \\ &+ \left((\alpha - L_s)v_n^* \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right) \end{aligned}$$

converges to zero when $n \rightarrow \infty$ which is obvious for the first two terms on the right-hand side and follows from

$$\left((\alpha - L_s)v_n^* \mid (\alpha - L_s)^{-1}(u_n - u_n^*) \right) = \left(v_n^* \mid (u_n - u_n^*) \right)$$

for the last term.

Altogether one obtains that

$$\begin{aligned}
& (u \mid G_\alpha u) \\
&= \lim_{n \rightarrow \infty} \left\{ 2 (u \mid (v_n + v_n^*)) - ((\alpha - L)(v_n + v_n^*) \mid (\alpha - L_s)^{-1}(\alpha - L)(v_n + v_n^*)) \right\} \\
&\leq \sup_{v \in D_0} \left\{ 2 (u \mid v) - ((\alpha - L)v \mid (\alpha - L_s)^{-1}(\alpha - L)v) \right\}
\end{aligned}$$

and it remains to show that

$$\sup_{v \in D_0} \left\{ 2 (u \mid v) - ((\alpha - L)v \mid (\alpha - L_s)^{-1}(\alpha - L)v) \right\} \leq (u \mid G_\alpha u). \quad (2.2)$$

Now choose $(v_n^*)_{n=1}^\infty \subseteq D_0$ such that

$$v_n^* \rightarrow G_\alpha^* u \quad \text{and} \quad (\alpha - L^*)v_n^* \rightarrow u \quad \text{in } H \text{ when } n \rightarrow \infty$$

and remark that $(\alpha - L_s)^{1/2}$ is well-defined on $D_0 \subseteq D(L_s)$. Then for every $v \in D_0$

$$\begin{aligned}
(u \mid v) &= (u \mid G_\alpha(\alpha - L)v) = (G_\alpha^* u \mid (\alpha - L)v) \\
&= \lim_{n \rightarrow \infty} (v_n^* \mid (\alpha - L_s)^{1/2}(\alpha - L_s)^{-1/2}(\alpha - L)v) \\
&= \lim_{n \rightarrow \infty} ((\alpha - L_s)^{1/2}v_n^* \mid (\alpha - L_s)^{-1/2}(\alpha - L)v) \\
&\leq \lim_{n \rightarrow \infty} \sqrt{(v_n^* \mid (\alpha - L_s)v_n^*) \cdot ((\alpha - L_s)v \mid (\alpha - L_s)^{-1}(\alpha - L)v)} \\
&\leq \lim_{n \rightarrow \infty} \left[(v_n^* \mid (\alpha - L_s)v_n^*) + ((\alpha - L_s)v \mid (\alpha - L_s)^{-1}(\alpha - L)v) \right] / 2 \\
&= \left[(u \mid G_\alpha u) + ((\alpha - L)v \mid (\alpha - L_s)^{-1}(\alpha - L)v) \right] / 2
\end{aligned}$$

since

$$(v_n^* \mid (\alpha - L_s)v_n^*) = (v_n^* \mid (\alpha - L^*)v_n^*) \longrightarrow (G_\alpha^* u \mid u) = (u \mid G_\alpha u), \quad n \rightarrow \infty,$$

and (2.2) follows. ■

Remark 2.5 For the proof of (2.2), D_0 only needs to be a core for $(L^*, D(L^*))$ but not for $(L, D(L))$. Of course $D_0 \subseteq D(L)$ must still be assumed.

Corollary 2.6 *If there exists $D_0 \subseteq H$ which is a core for $(L, D(L))$, $(L^*, D(L^*))$ and $(L_s, D(L_s))$ then*

$$(u \mid G_\alpha u) \leq (u \mid (\alpha - L_s)^{-1} u), \quad u \in H,$$

for every $\alpha > 0$.

Proof. If D_0 is a core for $(L, D(L))$, $(L^*, D(L^*))$ and $(L_s, D(L_s))$ then Lemma 2.4 implies

$$(u \mid G_\alpha u) \leq \sup_{v \in D_0} \left\{ 2 (u \mid v) - (v \mid (\alpha - L_s)v) \right\} = (u \mid (\alpha - L_s)^{-1} u).$$

Indeed

$$\begin{aligned} ((\alpha - L)v \mid (\alpha - L_s)^{-1}(\alpha - L)v) &= (v \mid (\alpha - L_s)v) + (L_a v \mid (\alpha - L_s)^{-1}L_a v) \\ &\geq (v \mid (\alpha - L_s)v) \end{aligned}$$

for $v \in D_0$ since $(\alpha - L_s)^{-1}$ is positive definite and finally one applies Lemma 2.4 in the special case $L = L_s$. \blacksquare

Remark 2.7 Not every perturbation (L, D_0) of a symmetric operator (S, D_0) has S as its symmetric part on D_0 . An easy example demonstrating this fact will be given in the next section, see Remark 3.1. So one has to be careful when checking the assumptions of Corollary 2.6.

Finally the impact of Corollary 2.6 on the situation described in Lemma 2.1 is discussed. So $H = L^2(\ell \otimes \nu)$, $(G_\alpha)_{\alpha>0}$ is the strongly continuous contraction resolvent associated with the right process $(s, \eta_s)_{s \geq 0}$ on H and $(L, D(L))$ is the generator of $(G_\alpha)_{\alpha>0}$. Denote by $(L^\eta, D(L^\eta))$ the generator of the strongly continuous contraction resolvent associated with the right process $(\eta_s)_{s \geq 0}$ on $L^2(\nu)$. Corollary 2.6 suggests to develop the inequality given in Lemma 2.1 by using

$$(\tilde{V} \mid G_{\frac{\beta}{2c}} \tilde{V})_{L^2(\ell \otimes \nu)} \leq (\tilde{V} \mid (\frac{\beta}{2c} - L_s)^{-1} \tilde{V})_{L^2(\ell \otimes \nu)}$$

and one wants to simplify $(\beta/(2c) - L_s)^{-1} \tilde{V}$ in this specific situation. Formally $L = \frac{\partial}{\partial s} + L^\eta$ and $L^* = -\delta_0(s) - \frac{\partial}{\partial s} + (L^\eta)^*$ where δ_0 denotes Dirac's delta function. So, under certain conditions on \tilde{V} , one should have an equality of the type

$$[(\alpha - L_s)^{-1} \tilde{V}](s, \eta) = [(\alpha - L^{\text{sym}})^{-1} \tilde{V}(s, \cdot)](\eta) \stackrel{\text{def}}{=} g(s, \eta)$$

where L^{sym} denotes the symmetric part of L^η in $L^2(\nu)$. The following lemma, first, lists sufficient conditions to ensure this and, second, presents the final bound on the left-hand side of the inequality in Lemma 2.1.

Lemma 2.8 (i) Fix $\alpha > 0$. If $g, L^\eta g, (L^\eta)^* g, \frac{\partial}{\partial s} g \in L^2(\ell \otimes \nu)$, $g(s, \cdot) \in D(L^\eta) \cap D((L^\eta)^*)$, $s \geq 0$, and $g(\cdot, \eta) \in C_0^1([0, \infty))$, $\eta \in X$, then $(\alpha - L_s)^{-1} \tilde{V} = g$.

(ii) Fix $\beta > 0$, $T > 0$, $c > 0$ and assume that there exists $D_0 \subseteq L^2(\nu)$ which is a core for $(L^\eta, D(L^\eta))$, $((L^\eta)^*, D((L^\eta)^*))$ and $(L^{\text{sym}}, D(L^{\text{sym}}))$. Then:

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V(s, \eta_{cs}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta c} \int_0^T (V(s, \cdot) \mid (\frac{\beta}{2c} - L^{\text{sym}})^{-1} V(s, \cdot))_{L^2(\nu)} ds$$

for all bounded measurable functions V on $[0, \infty) \times X$.

Proof. The conditions given in (i) are obvious conditions for $L_s g(s, \eta) = [L^{\text{sym}} g(s, \cdot)](\eta)$ to be true which indeed proves the claim.

Part (ii) only needs to be proven for $V(s, \eta) \mathbf{1}_{[0, T]}(s)$ instead of $V(s, \eta)$. The method is to approximate $V \mathbf{1}_{[0, T]}$ by 'good' functions V_n such that the functions g_n corresponding to

\tilde{V}_n satisfy the conditions of part (i). Going through the proof of Corollary 2.6 but using the conditions of part (i) reveals that one *can* conclude that

$$(\tilde{V}_n | G_{\frac{\beta}{2c}} \tilde{V}_n)_{L^2(\ell \otimes \nu)} \leq (\tilde{V}_n | (\frac{\beta}{2c} - L_s)^{-1} \tilde{V}_n)_{L^2(\ell \otimes \nu)} \quad (2.3)$$

in this specific case where, by part (i), the right-hand side is equal to

$$\begin{aligned} & \int_0^\infty (\tilde{V}_n(s, \cdot) | (\frac{\beta}{2c} - L^{\text{sym}})^{-1} \tilde{V}_n(s, \cdot))_{L^2(\nu)} ds \\ &= c \int_0^\infty e^{-\beta s} (V_n(s, \cdot) | (\frac{\beta}{2c} - L^{\text{sym}})^{-1} V_n(s, \cdot))_{L^2(\nu)} ds \end{aligned}$$

so that

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V_n(s, \eta_{cs}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta c} \int_0^\infty (V_n(s, \cdot) | (\frac{\beta}{2c} - L^{\text{sym}})^{-1} V_n(s, \cdot))_{L^2(\nu)} ds \quad (2.4)$$

for all n by Lemma 2.1. Taking limits when n goes to infinity in the above inequality finally proves part (ii). Notice that the upper limit of the ds -integration can indeed be changed to T because V_n approximates $V \mathbf{1}_{[0, T]}$. ■

Remark 2.9 (i) Taking limits in (2.4) makes clear that the approximation of $V \mathbf{1}_{[0, T]}$ by ‘good’ functions V_n is a standard approximation of a function in $L^2(\ell \otimes \nu)$ by in some sense ‘smooth’ functions and the details are therefore omitted.

(ii) Remember that the left-hand side of (2.3) can be transformed into

$$\int_0^\infty ds \int_0^\infty dr e^{-\beta(s+r)/c} \mathbf{E}_\nu V_n(s/c, \eta_s) V_n((s+r)/c, \eta_{s+r})$$

as in the proof of Lemma 2.1. However, it is not possible to transform the right-hand side of (2.3) into a similar expression because the resolvent $((\alpha - L_s)^{-1})_{\alpha > 0}$ is not associated with a process of the form $(s, \eta_s^{\text{sym}})_{s \geq 0}$.

(iii) In the case where V is time independent one can directly apply Corollary 2.6 to the right-hand side of the inequality in Remark 2.2(i) which gives

$$\int_0^T dt \mathbf{E}_\nu \left[\int_0^t V(\eta_{cs}) ds \right]^2 \leq \frac{2e^{\beta T}}{\beta^2 c} (V | (\beta/c - L^{\text{sym}})^{-1} V)_{L^2(\nu)}$$

for all bounded V in $L^2(\nu)$ hence for all $V \in L^2(\nu)$ by approximation. Notice that ν is an invariant measure for the resolvent $((\alpha - L^{\text{sym}})^{-1})_{\alpha > 0}$ under the conditions made.

3 Application to 1-dimensional Simple Exclusion

Fix $p, q \geq 0$ such that $p + q = 1$ and let $(\Omega, \mathcal{F}, \mathbf{P}_\eta, \eta \in \{0, 1\}^\mathbb{Z}, (\eta_t)_{t \geq 0})$ denote the strong Markov Feller process whose generator L acts on local functions $f : \{0, 1\}^\mathbb{Z} \rightarrow \mathbb{R}$ as

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} (2p \eta(x)(1 - \eta(x+1))[f(\eta^{x, x+1}) - f(\eta)] + 2q \eta(x)(1 - \eta(x-1))[f(\eta^{x, x-1}) - f(\eta)]) \quad (3.1)$$

where the operation

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & : z \neq x, y \\ \eta(x) & : z = y \\ \eta(y) & : z = x \end{cases}$$

exchanges the “spins” at x and y . This process is called simple exclusion process, see [L1999] for a good account on the existing theory.

Denote by $\nu_{1/2}$ the Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}}$ satisfying $\nu_{1/2}(\eta(x) = 1) = 1/2$ for all $x \in \mathbb{Z}$ which is one of the invariant ergodic states of the simple exclusion process. If

$$\mathbf{P} = \int \mathbf{P}_\eta d\nu_{1/2}(\eta) \quad \text{as well as} \quad \xi_t(x) = \frac{\eta_t(x) - \mathbf{E}\eta_t(x)}{\sqrt{\mathbf{Var}(\eta_t(x))}}$$

where \mathbf{E} and \mathbf{Var} stand for expectation and variance with respect to \mathbf{P} , respectively, then the process $(\xi_t)_{t \geq 0}$ is a stationary process on $(\Omega, \mathcal{F}, \mathbf{P})$ which takes values in $\{-1, 1\}^{\mathbb{Z}}$ and the push forward of $\nu_{1/2}$ with respect to the map

$$\eta \mapsto \xi \quad \text{given by} \quad \xi(x) = \frac{\eta(x) - 1/2}{\sqrt{1/4}}, \quad x \in \mathbb{Z},$$

is the invariant distribution of ξ_t , $t \geq 0$.

For $\Lambda \subseteq \mathbb{Z}$ finite, set $\xi_\Lambda = \prod_{x \in \Lambda} \xi(x)$ if Λ is not empty and $\xi_\emptyset = 1$ otherwise. Then, $\{\xi_\Lambda : \Lambda \subseteq \mathbb{Z} \text{ finite}\}$ forms an orthonormal basis of $L^2(\nu_{1/2})$. Hence the linear hull $\mathbf{Lin}\{\xi_\Lambda\}$ of $\{\xi_\Lambda : \Lambda \subseteq \mathbb{Z} \text{ finite}\}$ is dense in $L^2(\nu_{1/2})$. Remark that the operator $(L, \mathbf{Lin}\{\xi_\Lambda\})$ is closable on $L^2(\nu_{1/2})$ and that its closure $(L, D(L))$ generates a Markovian strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on $L^2(\nu_{1/2})$ which is associated with the transition semigroup of the strong Markov process $(\Omega, \mathcal{F}, \mathbf{P}_\eta, \eta \in \{0, 1\}^{\mathbb{Z}}, (\eta_t)_{t \geq 0})$. Furthermore, it follows from (3.1) that

$$L\xi_\Lambda = \gamma \mathcal{A}_+ \xi_\Lambda - \gamma \mathcal{A}_+^* \xi_\Lambda + \mathcal{S} \xi_\Lambda \quad \text{where} \quad \gamma = p - q$$

and

$$\mathcal{A}_+ \xi_\Lambda = \sum_{x \in \Lambda} [\mathbf{1}_{\Lambda^c}(x+1) \xi_{\Lambda \cup \{x+1\}} - \mathbf{1}_{\Lambda^c}(x-1) \xi_{\Lambda \cup \{x-1\}}].$$

The adjoint operator of $(\mathcal{A}_+, \mathbf{Lin}\{\xi_\Lambda\})$ with respect to the inner product on $L^2(\nu_{1/2})$ is denoted by \mathcal{A}_+^* . Its domain includes $\mathbf{Lin}\{\xi_\Lambda\}$ and, on this subdomain, it is given by

$$\mathcal{A}_+^* \xi_\Lambda = \sum_{x \in \Lambda} [\mathbf{1}_{\Lambda^c}(x+1) - \mathbf{1}_{\Lambda^c}(x-1)] \xi_{\Lambda \setminus \{x\}}.$$

The operator $(\mathcal{S}, \mathbf{Lin}\{\xi_\Lambda\})$ is a symmetric operator on $L^2(\nu_{1/2})$ satisfying

$$\mathcal{S} \xi_\Lambda = \mathcal{S}_0 \xi_\Lambda - 2|\Lambda| \xi_\Lambda$$

where

$$\begin{aligned} \mathcal{S}_0 \xi_\Lambda = \sum_{x \in \Lambda} & [\mathbf{1}_{\Lambda^c}(x+1) \xi_{\Lambda \setminus \{x\} \cup \{x+1\}} + \mathbf{1}_\Lambda(x+1) \xi_\Lambda \\ & + \mathbf{1}_{\Lambda^c}(x-1) \xi_{\Lambda \setminus \{x\} \cup \{x-1\}} + \mathbf{1}_\Lambda(x-1) \xi_\Lambda] \end{aligned}$$

and $|\Lambda|$ denotes the cardinality of Λ .

As a consequence, the adjoint $(L^*, D(L^*))$ of $(L, D(L))$ satisfies

$$L^* = \gamma \mathcal{A}_+^* - \gamma \mathcal{A}_+ + \mathcal{S} \quad \text{on} \quad \mathbf{Lin}\{\xi_\Lambda\}$$

and $(L^*, D(L^*))$ is the closure of $(L^*, \mathbf{Lin}\{\xi_\Lambda\})$. Thus if

$$G_\alpha = \int_0^\infty e^{-\alpha t} T_t dt, \quad \alpha > 0,$$

is the Markovian strongly continuous contraction resolvent generated by $(L, D(L))$ then $(L^*, D(L^*))$ generates G_α^* hence the condition (A1) in Section 2 is satisfied. Furthermore

$$L_s = \frac{L + L^*}{2} = S \quad \text{on} \quad \mathbf{Lin}\{\xi_\Lambda\}$$

and $(S, \mathbf{Lin}\{\xi_\Lambda\})$ is of course essentially self-adjoint, its closure being the generator of the symmetric simple exclusion process. Altogether the domain $\mathbf{Lin}\{\xi_\Lambda\}$ satisfies all the conditions imposed on the domain D_0 in Corollary 2.6.

Remark 3.1 The operator $(\gamma \mathcal{A}_+ + \mathcal{S}, \mathbf{Lin}\{\xi_\Lambda\})$ is an easy example of an operator which, when seen as a perturbation of $(\mathcal{S}, \mathbf{Lin}\{\xi_\Lambda\})$ does not have S as its symmetric part on $\mathbf{Lin}\{\xi_\Lambda\}$.

The example field of type (1.1) discussed in this paper is the density fluctuation field

$$Y_s^\varepsilon = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \xi_{s\varepsilon^{-2}}(x) \delta_{\varepsilon x}, \quad s \geq 0,$$

in the case of $\sqrt{\varepsilon}$ -asymmetric one-dimensional simple exclusion where p and q hence γ depend on ε such that

$$\gamma = \gamma_\varepsilon = \tilde{\gamma} \cdot \sqrt{\varepsilon}.$$

As a consequence L, T_t, G_α and \mathbf{E} introduced above are denoted by $L_\varepsilon, T_t^\varepsilon, G_\alpha^\varepsilon$ and \mathbf{E}_ε in what follows.

If ε is small then the field has a related approximate equation of type (1.2) with \mathcal{A} being the one-dimensional Laplacian and with $V_\varepsilon^G(s, \cdot)$ being of the special form $\tilde{\gamma} V_\varepsilon^G$ where

$$V_\varepsilon^G(\xi) = - \sum_{x \in \mathbb{Z}} G'(\varepsilon x) \xi(x) \xi(x+1).$$

Functions of this type are called quadratic fluctuations in this paper, *quadratic* since its summands are of type ξ_Λ with $|\Lambda| = 2$ and *fluctuations* since ξ is $(\eta - 1/2)/\sqrt{1/4}$. Remark that V_ε^G does not depend on s because of the special choice of $\nu_{1/2}$ to be the invariant state of the system. However, the choice of a different invariant state would only make the notation more complicated but would not affect the arguments used below.

Furthermore, V_ε^G remains bounded if G is a test function in the Schwartz space $\mathcal{S}(\mathbb{R})$ and this space of test functions is chosen in what follows.

The functional F used to replace

$$\int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \quad \text{by} \quad \int_0^t F(Y_s^\varepsilon, G) ds$$

depends on a further parameter N , that is $F = F_N$ in this example. It can be defined as follows. Fix an even non-negative test function d satisfying

$$\text{supp } d = [-1, 1] \quad \text{and} \quad \int_{\mathbb{R}} d(u) du = 1$$

and denote by d_N the function $x \mapsto Nd(Nx)$, $N \geq 1$. Remark that the convolution $Y_s^\varepsilon \star d_N$ is a C^∞ -function on \mathbb{R} satisfying

$$\int_{\mathbb{R}} H(u) dY_s^\varepsilon(u) = \lim_{N \uparrow \infty} \int_{\mathbb{R}} H(u) (Y_s^\varepsilon \star d_N)(u) du$$

for all $s \geq 0$ and all continuous functions H on \mathbb{R} with sufficiently fast decaying tails. Now set

$$F_N(\mathcal{Y}, G) = - \int_{\mathbb{R}} G'(u) (\mathcal{Y} \star d_N)^2(u) du, \quad \mathcal{Y} \in \mathcal{S}'(\mathbb{R}).$$

Fixing a finite time horizon $T > 0$, one wants to estimate

$$\int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t F_N(Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \right)^2$$

for small ε and large N . So fix ε, N and observe that

$$\begin{aligned} & \int_0^t F_N(Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \\ &= - \int_0^t \int_{\mathbb{R}} G'(u) (Y_s^\varepsilon \star d_N)^2(u) du ds + \int_0^t \sum_{x \in \mathbb{Z}} G'(\varepsilon x) \xi_{s\varepsilon^{-2}}(x) \xi_{s\varepsilon^{-2}}(x+1) ds \end{aligned}$$

hence, setting $H = -G'$, one can split

$$\int_0^t F_N(Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \quad \text{into} \quad \sum_{i=1}^4 \int_0^t V_{\varepsilon, N}^{H, i}(\xi_{s\varepsilon^{-2}}) ds$$

using further quadratic fluctuations $V_{\varepsilon, N}^{H, 1}, V_{\varepsilon, N}^{H, 2}, V_{\varepsilon, N}^{H, 3}, V_{\varepsilon, N}^{H, 4}$ given by

$$\begin{aligned} V_{\varepsilon, N}^{H, 1}(\xi) &= \sum_{x \in \mathbb{Z}} \int_{\mathbb{R}} [H(u) - H(\varepsilon x)] d_N(u - \varepsilon x) \sum_{\tilde{x} \in \mathbb{Z}} \varepsilon d_N(u - \varepsilon \tilde{x}) du \xi(x) \xi(\tilde{x}), \\ V_{\varepsilon, N}^{H, 2}(\xi) &= \varepsilon \sum_{x \in \mathbb{Z}} H(\varepsilon x) \int_{\mathbb{R}} d_N^2(u - \varepsilon x) du \xi(x) [\xi(x) - \xi(x+1)], \\ V_{\varepsilon, N}^{H, 3}(\xi) &= \varepsilon \sum_{x \neq \tilde{x}} H(\varepsilon x) \int_{\mathbb{R}} d_N(u - \varepsilon x) d_N(u - \varepsilon \tilde{x}) du \xi(x) [\xi(\tilde{x}) - \xi(x+1)], \\ V_{\varepsilon, N}^{H, 4}(\xi) &= \sum_{x \in \mathbb{Z}} H(\varepsilon x) \int_{\mathbb{R}} d_N(u - \varepsilon x) \left[\sum_{\tilde{x} \in \mathbb{Z}} \varepsilon d_N(u - \varepsilon \tilde{x}) - 1 \right] du \xi(x) \xi(x+1). \end{aligned}$$

Hence, if $(\cdot | \cdot)$ denotes the inner product on $L^2(\nu_{1/2})$ then

$$\begin{aligned} & \mathbf{E}_\varepsilon \left(\int_0^t F_N(Y_s^\varepsilon, G) \, ds - \int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) \, ds \right)^2 \\ & \leq 4 \sum_{i=1}^4 \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon,N}^{H,i}(\xi_{s\varepsilon^{-2}}) \, ds \right)^2 \end{aligned} \quad (3.2)$$

$$\leq 8 \sum_{i=1}^3 \varepsilon^4 \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon,N}^{H,i} | T_r^\varepsilon V_{\varepsilon,N}^{H,i}) + 4 \mathbf{E}_\varepsilon \left(\varepsilon^2 \int_0^{t\varepsilon^{-2}} V_{\varepsilon,N}^{H,4}(\xi_s) \, ds \right)^2 \quad (3.3)$$

for all $t \geq 0$ by the Markov property.

In what follows, $\|H\|_p$ denotes the norm of a test function H in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Then, as $|\sum_{\tilde{x} \in \mathbb{Z}} \varepsilon d_N(u - \varepsilon \tilde{x}) - 1| \leq \varepsilon^2 N^2 \|d''\|_\infty$ is easily realised, one obtains that

$$\mathbf{E}_\varepsilon \left(\varepsilon^2 \int_0^{t\varepsilon^{-2}} V_{\varepsilon,N}^{H,4}(\xi_s) \, ds \right)^2 \leq \varepsilon^2 N^4 \|d''\|_\infty^2 \|H\|_1^2 \cdot t^2, \quad t \geq 0. \quad (3.4)$$

The other three integrals in (3.3) are much harder to control. But the following lemma gives bounds for these integrals if T_r^ε is substituted by the semigroup T_r^{sym} associated with the symmetric simple exclusion process.

Lemma 3.2 *Let $H \in \mathcal{S}(\mathbb{R})$ be a test function such that $\int_{\mathbb{R}} H(u) \, du = 0$. Then*

$$\begin{aligned} (i) \quad & \varepsilon^4 \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon,N}^{H,1} | T_r^{\text{sym}} V_{\varepsilon,N}^{H,1}) \leq c_0 t^2 \left(\|d\|_2^4 \frac{\|(1+u^2)H''\|_\infty^2}{N^2} + \|d\|_\infty^2 \frac{\|(1+u^2)H'\|_\infty^2}{N} \right) \\ (ii) \quad & \varepsilon^4 \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon,N}^{H,2} | T_r^{\text{sym}} V_{\varepsilon,N}^{H,2}) \leq t^2 \|d\|_2^4 \left(\varepsilon^2 N^2 \|H'\|_\infty^2 + \varepsilon N^2 \|(1+u^2)H\|_\infty^2 \right) \\ (iii) \quad & \varepsilon^4 \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon,N}^{H,3} | T_r^{\text{sym}} V_{\varepsilon,N}^{H,3}) \leq c_0 t^2 \|d\|_\infty^2 \frac{\|(1+u^2)H\|_\infty^2}{N^{1/3}} \end{aligned}$$

for all $t \geq 0$, $N = 1, 2, \dots$, $\varepsilon > 0$ where c_0 is a constant which neither depends on the chosen test function H nor on the mollifier d .

Proof. For showing (i) one splits $V_{\varepsilon,N}^{H,1}$ into two sums

$$\begin{aligned} & \varepsilon \sum_{x \in \mathbb{Z}} \int_{\mathbb{R}} [H(u) - H(\varepsilon x)] d_N^2(u - \varepsilon x) \, du \\ & + \sum_{x \neq \tilde{x}} \int_{\mathbb{R}} [H(u) - H(\varepsilon x)] d_N(u - \varepsilon x) \varepsilon d_N(u - \varepsilon \tilde{x}) \, du \, \xi(x) \xi(\tilde{x}). \end{aligned} \quad (3.5)$$

Applying the Taylor expansion

$$H(u) - H(\varepsilon x) = (u - \varepsilon x) H'(\varepsilon x) + (u - \varepsilon x)^2 H''(\theta_{\varepsilon x}^u)/2 \quad \text{with} \quad \theta_{\varepsilon x}^u \in [\varepsilon x - u, \varepsilon x + u]$$

to the first sum yields

$$\left| \varepsilon \sum_{x \in \mathbb{Z}} \int_{\mathbb{R}} [H(u) - H(\varepsilon x)] d_N^2(u - \varepsilon x) du \right| \leq \varepsilon \sum_{x \in \mathbb{Z}} \int_{\mathbb{R}} \frac{(u - \varepsilon x)^2}{2} |H''(\theta_{\varepsilon x}^u)| d_N^2(u - \varepsilon x) du$$

where $\int_{\mathbb{R}} (u - \varepsilon x) d_N^2(u - \varepsilon x) du$ vanishes because the mollifier d is even. Now observe that

$$|H''(\theta_{\varepsilon x}^u)| \leq \begin{cases} \sup_{\tilde{u}} |(1 + \tilde{u}^2)H''(\tilde{u})| \cdot [1 + (\varepsilon x - \frac{1}{N})^2]^{-1} & : x > 1/\varepsilon \\ \sup_{\tilde{u}} |(1 + \tilde{u}^2)H''(\tilde{u})| \cdot [1 + (\varepsilon x + \frac{1}{N})^2]^{-1} & : x < -1/\varepsilon \end{cases}$$

on the set $\{u \in \mathbb{R} : d_N(u - \varepsilon x) \neq 0\}$ since $\text{supp } d_N = [-\frac{1}{N}, \frac{1}{N}]$. Furthermore

$$(u - \varepsilon x)^2 \leq 1/N^2 \quad \text{on} \quad \{u \in \mathbb{R} : d_N(u - \varepsilon x) \neq 0\}$$

and

$$\int_{\mathbb{R}} d_N^2(u - \varepsilon x) du = N \|d\|_2^2 \quad \text{for all } x \in \mathbb{Z}.$$

Therefore, the sum immediately above (3.5) can be estimated by

$$\begin{aligned} \frac{\|d\|_2^2}{2N} & \left(\|(1+u^2)H''\|_{\infty} \overbrace{\sum_{x < -\frac{1}{\varepsilon}} \varepsilon [1 + (\varepsilon x + \frac{1}{N})^2]^{-1}}^{\leq \pi/2} + 2\|H''\|_{\infty} + \|(1+u^2)H''\|_{\infty} \overbrace{\sum_{x > \frac{1}{\varepsilon}} \varepsilon [1 + (\varepsilon x - \frac{1}{N})^2]^{-1}}^{\leq \pi/2} \right) \\ & \leq \frac{\|d\|_2^2}{2N} (\pi + 2) \|(1+u^2)H''\|_{\infty} \end{aligned} \quad (3.6)$$

which explains the first summand on the right-hand side of (i).

The second summand is a bound of the integral on the left-hand side of (i) but with $V_{\varepsilon, N}^{H,1}$ replaced by the fluctuation field given by (3.5). This bound is obtained by copying the proof of Lemma 1 in [A2007] for $x_0 = 1$ using the equality

$$H(u) - H(\varepsilon x) = (u - \varepsilon x) H'(\tilde{\theta}_{\varepsilon x}^u)$$

where $\tilde{\theta}_{\varepsilon x}^u \in [\varepsilon x - u, \varepsilon x + u]$. The only difference to the proof of Lemma 1 in [A2007] is that, similar to how (3.6) was derived, the sum $\sum_{x \in \mathbb{Z}} \varepsilon |H'(\tilde{\theta}_{\varepsilon x}^u)|$ is estimated by $(\pi + 2) \sup_{\tilde{u}} |(1 + \tilde{u}^2)H'(\tilde{u})|$ and not by $c_H \|H'\|_{\infty}$.

The left-hand side of (ii) can be estimated the same way the sum $S_1(t, \varepsilon, N)$ was estimated in the proof of Theorem 1 in [A2007]. Following this proof would give

$$\varepsilon^4 \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon, N}^{H,2} | T_r^{\text{sym}} V_{\varepsilon, N}^{H,2}) \leq t^2 \|d\|_2^4 \left(N^2 \left(\sum_{x \in \mathbb{Z}} \varepsilon H(\varepsilon x) \right)^2 + \varepsilon N^2 c_H \|H\|_{\infty}^2 \right)$$

if H were a test function with compact support. But this implies (ii) because, by our assumption $\int_{\mathbb{R}} H(u) du = 0$, it holds that $|\sum_{x \in \mathbb{Z}} \varepsilon H(\varepsilon x)| \leq \varepsilon \|H'\|_{\infty}$ by our assumption $\int_{\mathbb{R}} H(u) du = 0$. Again, as in the proof of part (i) above, the bound $c_H \|H\|_{\infty}$ is replaced by $\|(1+u^2)H\|_{\infty}$.

Finally, the inequality (iii) can be established by copying the proof of Theorem 1 in [A2007] with respect to $S_{6-9}(t, \varepsilon, N)$ for $x_0 = 1$ and $\alpha = 2/3$ manipulating the constant c_H accordingly. \blacksquare

The next lemma is the key result of this section. It translates the estimates given by Lemma 3.2 into estimates of the summands in (3.2) on page 13 when integrating them against dt over $t \in [0, T]$.

Lemma 3.3 *Let $H \in \mathcal{S}(\mathbb{R})$ be a test function such that $\int_{\mathbb{R}} H(u) du = 0$ and fix a finite time horizon $T > 0$. Then*

$$\begin{aligned} (i) \quad & \int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon, N}^{H,1}(\xi_{s\varepsilon^{-2}}) ds \right)^2 \leq e^T C_d \left(\frac{\|(1+u^2)H''\|_\infty^2}{N^2} + \frac{\|(1+u^2)H'\|_\infty^2}{N} \right) \\ (ii) \quad & \int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon, N}^{H,2}(\xi_{s\varepsilon^{-2}}) ds \right)^2 \leq e^T C_d \left(\varepsilon^2 N^2 \|H'\|_\infty^2 + \varepsilon N^2 \|(1+u^2)H\|_\infty^2 \right) \\ (iii) \quad & \int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon, N}^{H,3}(\xi_{s\varepsilon^{-2}}) ds \right)^2 \leq e^T C_d \frac{\|(1+u^2)H\|_\infty^2}{N^{1/3}} \\ (iv) \quad & \int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon, N}^{H,4}(\xi_{s\varepsilon^{-2}}) ds \right)^2 \leq T^3 C_d \varepsilon^2 N^4 \|H\|_1^2 \end{aligned}$$

for all $N = 1, 2, \dots$, $\varepsilon > 0$ where C_d is a constant which only depends on the choice of the mollifier d .

Proof. Choose $\beta = 1$ and consider $i = 1, 2, 3$ first. Then, applying the inequality in Remark 2.9(iii) with respect to $c = \varepsilon^{-2}$, one obtains that

$$\int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t V_{\varepsilon, N}^{H,i}(\xi_{s\varepsilon^{-2}}) ds \right)^2 \leq 2e^T \varepsilon^2 (V_{\varepsilon, N}^{H,i} | (\varepsilon^2 - L^{\text{sym}})^{-1} V_{\varepsilon, N}^{H,i})$$

where

$$(V_{\varepsilon, N}^{H,i} | (\varepsilon^2 - L^{\text{sym}})^{-1} V_{\varepsilon, N}^{H,i}) = \varepsilon^2 \int_0^\infty dt e^{-t} \int_0^{t\varepsilon^{-2}} ds \int_0^s dr (V_{\varepsilon, N}^{H,i} | T_r^{\text{sym}} V_{\varepsilon, N}^{H,i})$$

by a calculation similar to the proof of Lemma 2.1 in the time independent case. Hence (i)-(iii) follows from Lemma 3.2 since $\int_0^\infty t^2 e^{-t} dt$ is finite.

Finally, (iv) follows directly from (3.4) by integration against dt over $t \in [0, T]$. \blacksquare

Applying Lemma 3.3 in the case where H is taken to be $-G'$ immediately gives the corollary below. Notice that $\int_{\mathbb{R}} H(u) du = 0$ is of course satisfied for $H = -G'$.

Corollary 3.4 *Fix an arbitrary but finite time horizon $T > 0$. Then, for every smooth test function $G \in \mathcal{S}(\mathbb{R})$, it holds that*

$$\lim_{N \uparrow \infty} \limsup_{\varepsilon \downarrow 0} \int_0^T dt \mathbf{E}_\varepsilon \left(\int_0^t F_N(Y_s^\varepsilon, G) ds - \int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \right)^2 = 0$$

where

$$F_N(\mathcal{Y}, G) = - \int_{\mathbb{R}} G'(u)(\mathcal{Y} \star d_N)^2(u) du \quad \text{and} \quad V_\varepsilon^G(\xi) = - \sum_{x \in \mathbb{Z}} G'(\varepsilon x) \xi(x) \xi(x+1)$$

for $\mathcal{Y} \in \mathcal{S}'(\mathbb{R})$ and $\xi \in \{-1, 1\}^{\mathbb{Z}}$, respectively.

Remark 3.5 A replacement of

$$\int_0^t V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) ds \quad \text{by} \quad \int_0^t F_N(Y_s^\varepsilon, G) ds$$

for every $t \in [0, T]$ and not only for an average over $t \in [0, T]$ was shown in [JG2010] for a slightly different functional F_N . However it has been shown in [A2012] that many of the conclusions drawn from the stronger replacement result² in [JG2010] can actually be obtained by only applying the weaker replacement result of Corollary 3.4.

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²This replacement result was called ‘Second-order Boltzmann-Gibbs principle’.